Poincaré's contribution to the development of the theory of canonical transformation[1]

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(Preparation) What is canonical transformation?

Before our historical examination, we will confirm the terminology related to canonical transformation.

Let us consider the transformation of variables that transform so-called canonical equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad H = H(t, q_1, \cdots, q_n, p_1, \cdots, p_n) \quad (1)$$

to other canonical equations

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}, \quad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i}, K = K(t, Q_1, \cdots, Q_n, P_1, \cdots, P_n). \quad (2)$$

In addition to a notably symmetrical form, canonical equations have important mathematical properties. It then is useful if one transforms equations by maintaining their canonical form. In this report, the term canonical transformation is used to describe the transformation of variables by maintaining the canonical form of the equations involved.

In modern textbooks, old variables $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ and new variables $(Q_1, \cdots, Q_n, P_1, \cdots, P_n)$ are shown as bound by equations with an appropriate function S :

$$\Sigma p_i \dot{q}_i - H = \Sigma P_i \dot{Q}_i - K + \frac{dS}{dt} \qquad \text{or} \quad dS = \Sigma(p_i dq_i - P_i dQ_i) + (K - H)dt \qquad (3)$$

where ・ means a derivative with respect to *t*, and the transformation become canonical. The function S is called a generating function. Since canonical equations (1) are derived from the so-called Hamilton's principle

$$\delta \int_{t_0}^{t_1} L(t, q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) dt = 0, \quad (4)$$

and equations (2) are drawn from principle (4), which is written as

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$(Q_1,\cdots,Q_n,\dot{Q}_1,\cdots,\dot{Q}_n)$, relation (3) is obtained. We adopt a complete solution of the Hamilton–Jacobi equation as the generating function:

$$\frac{\partial S}{\partial t}+H\left(q_1,\ldots,q_n,\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n},t\right)=0.$$

In some cases, we call canonical equations Hamiltonian equations, and we call functions in the equations, such as *H* and *K,* Hamiltonian functions. Today we often write equations of motion in canonical forms. In dynamics, q_i is generalized coordinates, p_i is generalized momentum, and *H* is total energy, which does not explicitly depend on time. In this report, we mainly discuss the case where *H* does not explicitly depend on time, and *H* is not changed after the transformation of variables; namely *H = K.* To distinguish them from time dependent case, In this report, time independent canonical equations corresponding to (1) and (2) are called (1') and (2').

1. Introduction

At the end of the nineteenth century and the beginning of the twentieth century, various influential textbooks on celestial mechanics were published. Examples are the following:

- E. W. Brown (1896) *An Introductory Treatise on the Lunar Theory.*
- F. F. Tisserand (1889-1896) *Traité de Mécanique Céleste*, 4 vols.
- H. Poincaré (1892-1899) *Les Méthodes Nouvelles de la Mécanique Céleste*, 3 vols.
- H. Poincaré (1905-1910) *Leçons de Mécanique Céleste*, 3 vols.
- C. V. L. Charlier (1902, 1907) *Die Mechanik des Himmels*, 2 vols.
- F. Moulton (1902), *An Introduction to Celestial Mechanics.*
- E.T. Whittaker (1904) *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.*

Some authors, such as Tisserrand, Poincaré, Charlier, and Whittaker, used canonical equations rather than Newtonian equations of motion in discussing the mathematical properties of canonical equations. Among these works, Poincaré's two series of books are notable. The idea of canonical transformation was introduced at the beginning, whereas in other books, it was introduced in the middle or latter parts. This fact suggests that Poincaré paid particular attention to canonical transformation.

This report examines Poincaré's articles that were published around 1900, which include discussions related to canonical transformation and, which are listed chronologically as follows:

(1) "Sur la problème des trois corps et les équation de la dynamique," *Acta Mathematica*, 13, 1890, pp.1-270.
(2) *Les Méthodes Nouvelles*, Vol.1, 1892.
(3) "Sur une forme nouvelle des équations du problème des trois corps," *Comptes rendus* , 123, 1896, pp.1031-1035.
(4) "Sur une forme nouvelle des équations du problème des trois corps," *Bulletin Astronomique*, 14 ,1897, pp.53-67, *Acta Mathematica*, 21, pp.83-97.
(5) *Les Méthodes Nouvelles*, Vol.3, 1899.
(6) *Leçons de Mécanique Céleste*, Vol.1, 1905.

In the following section, we will clarify Poincaré's essential contribution to constructing the modern theory of canonical transformation.

2. A Brief History of Canonical Equations

2.1. Jacobi's general theory of canonical transformation

First, we outline the history of canonical equations before 1890. The canonical form of equations originated in Poisson's 1809 paper,[3] in which he wrote the equation of perturbation function as follows:

$$\frac{d\alpha_i}{dt} = \frac{\partial \Omega}{\partial \alpha_i'}, \quad \frac{d\alpha_i'}{dt} = -\frac{\partial \Omega}{\partial \alpha_i} \quad (i = 1, \ldots, n) \quad (2.1\text{-}1)$$

where $\Omega = \Omega(t, \alpha_1, \ldots, \alpha_n, \alpha_1', \ldots, \alpha_n')$ is the perturbation function, $(\alpha_1, \ldots, \alpha_n)$ are initial positions, and $(\alpha_1', \ldots, \alpha_n')$ are initial velocities. Lagrange

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[3] S. D. Poisson, " Mémoire sur la Variation des Constantes arbitraires dans les Questions de la Mécanique," Journal de l'École Polytechnique 8 (15 cahier), 1809, pp.266-344.

introduced these equations in the second version of his famous *Mécanique analytique*.[4] W. R. Hamilton noted Poisson and Lagrange's forms and derived equations of motions in the form of (1') in 1835.[5] He applied his original dynamical theory to what he called the "Problem of Perturbation."

In 1837, Jacobi named equations that have a symmetrical form "canonical".[6] He seemed to have the idea that the transformation of variables in which the canonical form of equations is maintained. He derived the following theorem, which was published in 1837. However, its proof was published after his death.[7]

Jacobi's second theorem

The canonical form of equations is preserved if the old and new variables are related as follows:

$$\frac{\partial \psi}{\partial P_i} = Q_i \quad \frac{\partial \psi}{\partial p_i} = -q_i \quad (i = 1, \cdots, n) \quad (2.1\text{–}2)$$

where $\psi = \psi(P_1, \cdots, P_n, p_1, \cdots, p_n)$

In this case, the Hamiltonian function is also preserved, namely $K = H$ in equations (2'). We call function ψ a generating function in accordance with modern terminology. Except equation (2.1-2), Jacobi did not give any information about ψ. However, the function ψ is tended to be confused with the complete solution of the Hamilton–Jacobi equation associated with the original canonical equations. The other Jacobi's theorem, which was also derived around 1837, presumably caused such confusion.

Jacobi's first theorem[8]

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[4] J. L. Lagrange, M*écanique analytique* (Second edition) Vol.1 (1811) ,Vol.2 (1815). Reproduced in Œuvres Vol. 11, Vo.12.

[5] W. R. Hamilton, "Second Essay on a General Method in Dynamics", *Philosophical Transaction of the Royal Society*, Part 1, 1835, pp.95-144. = in *the Mathematical Papers of Sir William Rowan Hamilton*, vol.2, pp.212- 216.

[6] C.G.J. Jacobi, "Note sur l'intégration des équations différentielles de la dynamique", *Comptes Rendus* 5, 1837 pp.61-67.= in *C.G.J. Jacobi's Gesammelte Werke*, Vol. 4, pp.131-136.

[7] C.G.J. Jacobi, "Über diejenigen Probleme der Mechanik in welchem eine Kräftefunction existirt und über die Theorie der Störungen ," =in *Werke* Vol. 5, pp.219-395.

[8] Jacobi showed his idea in the framework of Newtonian mechanics in "Über die Reduction der Integration der partiellen Differentialgleichungen

If a complete solution

$$S = S(t, q_1, \cdots, q_n, \alpha_1, \cdots, \alpha_n)$$

(where $\alpha_1, \cdots, \alpha_n$ are arbitrary constants) of the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(t, q_1, \cdots, q_n, \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_n}\right) = 0$$

are obtained, solutions to canonical equations (1) are given by

$$\frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial \alpha_i} = \beta_i \qquad (i = 1, \cdots, n) \qquad (2.1\text{-}3)$$

where $\beta_1, \cdots, \beta_n$ are new arbitrary constants.

Jacobi's first theorem states that solutions to canonical equations, which are a system of ordinary differential equations, are reduced to those of partial differential equations. Although the statements of Jacobi's two theorems are different, both involve the properties of canonical equations. Furthermore, according to modern knowledge, a complete solution of the Hamilton–Jacobi equation $S = S(q_1, \cdots q_n, \alpha_1, \cdots, \alpha_n)$ works as the generating function that transforms old variables $(q_1, \cdots q_n, p_1, \cdots p_n)$ to new ones $(\beta_1, \cdots, \beta_n, \alpha_1, \cdots, \alpha_n)$. Hence, it is not surprising that function ψ is confused with the complete solution. As we will see, Poincaré confused them in 1892. In accordance with Poincaré's numbering, in this report, we consider Jacobi's first theorem that in which the reduction of the solution to the canonical equation is the partial differential equation. We consider Jacobi's second theorem that in which the condition of the canonical transformation

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erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systems gewohnlicher Differentialgleichungen", *Journal für die reine und angewandte Mathematik*, 17, 1837, pp.97-162.=in Werke, vol.4, pp.57-127. He developed this theorem in 1842 and 1843, which was edited and published by Clebsch in *Vorlesungen über Dynamik* in 1866: *Werke Supplementband*.

is given.

2.2 Canonical equations in the middle of the nineteenth century

In the middle of the nineteenth century, mathematicians did not develop Jacobi's general theory of canonical transformation. However, we find a kind of canonical transformation in the procedure they used to reduce the degree of freedom in the three-body problem. Whittaker's report of 1899 includes examples by Bour (1856), Scheibner (1866, 1868), Mathieu (1874), and so on. We now examine Whittaker's description of Schreiber's reduction.

Scheibner set q_1, q_2, q_3 to be the mutual distances of three bodies, $p_1 = \frac{\partial T}{\partial q_1}$, $p_2 = \frac{\partial T}{\partial q_2}$, $p_3 = \frac{\partial T}{\partial q_3}$ where T is kinetic energy, q_4 is the angle that the node (of the plane of the bodies on the invariant plane) makes with one of the principal axes of the bodies at their center of gravity, and set $p_4 = k\cos i$, where k is the constant of angular momentum on the invariant plane, and i is the angle between the plane of the bodies and the invariable plane. Through these transformations of variables, the canonical equations in the three-body problem are reduced to

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, 3, 4)$$

where H is a certain function of q_i and p_i, and $H = \text{const}$ is an integral of the system. Whittaker's main interest is Scheibner's success in reducing the number of degrees in the three-body problem from 18 to 8. However, from our perspective, Scheibner performed the canonical transformation of variables for particular problem.

The contemporary mathematicians tried to find a way to reduce their problem independently of the general transformation theory. Moreover, their accumulated thought influenced Poincaré's consideration of this problem.

2.3 Delaunay's equation (1860)[9]

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[9] C. E. Delaunay, "Théorie du mouvement de la lune I", *Mémoire de*

In 1860, Charles Delaunay derived his original canonical equations of perturbation function F as follows:

$$\frac{dL}{dt}=\frac{\partial F}{\partial l},\frac{dG}{dt}=\frac{\partial F}{\partial g},\frac{d\Theta}{dt}=\frac{\partial F}{\partial \theta},$$
$$\frac{dl}{dt}=-\frac{\partial F}{\partial L},\frac{dg}{dt}=-\frac{\partial F}{\partial G},\frac{d\theta}{dt}=-\frac{\partial F}{\partial \Theta}$$

where a is the major axis, e is eccentricity, i is inclination, l is the mean anomaly, g is the longitude of the ascending node; $g+\theta$: mean anomaly, $L=\sqrt{\mu a}, G=L\sqrt{a(1-e^2)}, \Theta=G\cos i$, μ : sum of three masses.

These canonical equations, which involve orbital elements, worked effectively in his lunar theory. and they are included in modern textbooks of celestial mechanics. The readers of these textbooks may infer that Delaunay transformed the canonical equations of motion and derived his equations. However, Delaunay, while noting Poisson–Lagrange's equations of the perturbation function, derived these equations from Newtonian equations of motion, which are independent of the concept of canonical transformation. In any case, the appearance of new canonical equations led to the general theory of canonical transformation.

3. Poincaré's Consideration of the Canonical Transformation

3.1 Poincaré's description of the canonical transformation in 1890

To the best of my knowledge, Poincaré's first note on canonical transformation is included in his famous 1890 paper on the three-body problem. Section 4 in the second part of this paper was devoted to studying a case with only two degrees of freedom. Poincaré transformed Tisserand's equations of the perturbation function, which are Delaunay's equations in the two-dimensional case,

$$\begin{cases} \frac{dL}{dt}=\frac{\partial R}{\partial l}, & \frac{dl}{dt}=-\frac{\partial R}{\partial L} \\ \frac{dG}{dt}=\frac{\partial R}{\partial g}, & \frac{dg}{dt}=-\frac{\partial R}{\partial G} \end{cases}$$

to those of the canonical equation of motion,

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l'Academie des Sciences, 28, 1860, pp.1-883.

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad (i=1,2) \;\; H = H(q_1, q_2, p_1, p_2)$$

by setting (1890, pp.170–171)

$$q_1 = G, q_2 = L, p_1 = g - t, p_2 = l.$$

Poincaré also noted that if old variables (q_1, q_2, p_1, p_2) and new variables (Q_1, Q_2, P_1, P_2) are related by

$$Q_1 = q_1 + q_2, \quad Q_2 = q_1 - q_2, \quad P_1 = \frac{1}{2}(p_1 + p_2), \quad P_2 = \frac{1}{2}(p_1 - p_2)$$

Then the canonical form of equations is preserved (1890, p. 175).

Poincaré provided very simple examples of the two-dimensional case. However, these two examples have different aspects: the first relates Delaunay's equations to equations of motion; the second is an example of the linear transformation of variables that retain the canonical form of equations. Poincaré seemed to be seeking a general theory of canonical transformation. However, there is no evidence that indicates whether Poincaré noted Jacobi's second theorem or not at that time.

3.2 Poincaré's attitude in 1892

Poincaré's famous book, *Les Méthodes Nouvelles* (vol. 1, 1892) begins by introducing canonical equations of motion that were derived from Newtonian equations of motion. Next, Poincaré introduced Jacobi's first theorem without proof. Subsequently, he mentioned what he called Jacobi's second theorem without proof. Poincaré's version of Jacobi's second theorem is differs slightly from Jacobi's original theorem. According to Poincaré, an arbitrary function $S(p_1, \ldots, p_n, Q_1, \ldots, Q_n)$ leads the canonical transformation if the following relations hold:

$$q_i = \frac{\partial S}{\partial p_i}, \quad P_i = \frac{\partial S}{\partial Q_i} \quad (i = 1, \ldots, n) \quad (3.2\text{-}1)$$

In the modern view, minus signs should be put in front of these two equations.

However, Poincaré's developed his idea based on a serious misunderstanding. Neither Jacobi nor Poincaré mentioned that function S

is a complete solution of the Hamilton–Jacobi equation. Instead, Poincaré adopted the complete solution as a generating function and derived Delaunay's equations of perturbation function from the canonical equation of motion. He believed that he succeeded in transforming the equation of motion by deriving it from Delaunay's equation. However, his procedure has some gaps and cannot be followed. Subsequently, Poincaré found this weakness in his argument, and his revision of Jacobi's theorems was published in 1905.

3.3 Poincaré's understanding of the canonical transformation in 1896–1897

In his two papers entitled "Sur une forme nouvelle des équations du problème des trois corps," Poincaré described his new original properties of canonical transformation. Because the 1897 paper developed the idea demonstrated in the 1896 paper, we focus on the former.

Similar to other mathematicians in the mid-nineteenth century, Poincaré sought the transformation of variables that would reduce the degrees of freedom in three-body problems. He tried to find transformations that met the following conditions; linear transformations that retain the canonical form of equations and equations of areas. He set the three-body problem as follows.

Let *A, B,* and *C* be three masses of $m_1(=m_2=m_3)$, $m_4(=m_5=m_6)$ $m_7(=m_8=m_9)$ whose generalized coordinates are $A(q_1,q_2,q_3)$, $B(q_4,q_5,q_6)$, $C(q_7,q_8,q_9)$ and $p_i=m_i\dfrac{dx_i}{dt}$. Then the canonical equations of motion (1'), which is composed of 18 equations, describes the three-body problem. Poincaré noted that the canonical form is preserved if there are relationships between the old and new variables hold without proof:

$$\sum Q_i dP_i - \sum q_i dp_i = exact. \quad (3.3\text{-}1)$$

After he mentioned the conditions that keep equations of areas without proof,[10] he provided two examples of the transformations.

[10] Poincaré's conditions of preserving the equation of area are
(1) $Q_{3k}, Q_{3k+1}, Q_{3k+2}$ depend on only $q_{3k}, q_{3k+1}, q_{3k+2}$, while $P_{3k}, P_{3k+1}, P_{3k+2}$ only $p_{3k}, p_{3k+1}, p_{3k+2}$,

Poincaré's transformation (α)

$$p_1 = P_1,\ p_4 = P_4,\ q_7 = Q_7,\ q_1 - q_7 = Q_1,\ q_4 - q_7 = Q_4,\ P_7 = p_1 + p_4 + p_7$$

(Similar relations hold for $q_2, q_3, q_5, q_6, q_8, q_9$.)

Poincaré's transformation (β)

Let $G(Q_7, Q_8, Q_9)$ be the center of gravity of the three bodies, $D(\xi_1, \xi_2, \xi_3)$ be that of A and C,

$$Q_1 = q_1 - q_7,\ Q_2 = q_2 - q_8,\ Q_3 = q_3 - q_9,\ Q_4 = q_4 - \xi_1,\ Q_5 = q_5 - \xi_2,\ Q_6 = q_6 - \xi_3$$

$$m_1' = m_2' = m_3' = \frac{m_1 m_2}{m_1 + m_7},\ m_4' = m_5' = m_6' = \frac{m_4(m_1 + m_7)}{m_1 + m_4 + m_7},\ m_7' = m_8' = m_9' = m_1 + m_4 + m_7$$

and $P_i = m_i' \dfrac{dQ_i}{dt}$ where m_i' are masses after transformation.

After he reduced the degrees of freedom from 18 to 12 while preserving the canonical form of equations through these two transformations, Poincaré performed a further reduction. He transformed (Q_i, P_i) $(i = 1, \ldots, 6)$ to $L, G, \Theta, l, g, \theta$ and $L', G', \Theta', l', g', \theta'$, which are the variables that appear in Delaunay's equations. He defined transformation as follows:

$$\begin{cases} Q_k = \varphi_k(L, G, \Theta, l, g, \theta), \qquad P_k = \dfrac{\beta}{L^3}\dfrac{d\varphi_k}{dl} \quad (k = 1, 2, 3) \\ \\ Q_k = \varphi_k(L', G', \Theta', l', g', \theta'), \qquad P_k = \dfrac{\beta'}{L^3}\dfrac{d\varphi_k}{dl'} \quad (k = 4, 5, 6) \end{cases} \tag{3.3-2}$$

where β, β' are constants.

He then stated that the canonical form of equations is preserved if

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(2) The same relations between $Q_{3k}, Q_{3k+1}, Q_{3k+2}$ and $q_{3k}, q_{3k+1}, q_{3k+2}$ hold for all k,

(3) The same relations between $P_{3k}, P_{3k+1}, P_{3k+2}$ and $p_{3k}, p_{3k+1}, p_{3k+2}$ hold for all k,

where $k = 0, 1, 2$.

$$P_1dQ_1 + P_2dQ_2 + P_3dQ_3 - \beta(Ldl + Gdg + \Theta d\theta) \quad (3.3\text{-}3a)$$

$$P_4dQ_4 + P_5dQ_5 + P_6dQ_6 - \beta'(L'dl' + G'dg' + \Theta' d\theta') \quad (3.3\text{-}3b)$$

are exact without proof. He concluded the transformations defined by (3.3-2) retain the canonical form of equations. Equations (1') are finally transformed to

$$\begin{cases} \dfrac{dl}{dt} = \dfrac{\partial H}{\beta dL} & \dfrac{dL}{dt} = -\dfrac{\partial H}{\beta dl} \\[2em] \dfrac{dl'}{dt} = \dfrac{\partial H}{\beta' dL'} & \dfrac{dL'}{dt} = -\dfrac{\partial H}{\beta' dl'} \end{cases}$$

If we write old variables as (q_i, p_i) and new variables as (Q_i, P_i), and we neglect constants ß and ß', equations (3-2.3ab) are written as

$$\sum P_i dQ_i - \sum p_i dq_i = exact. \quad (3.3\text{-}4)$$

which are the modern relationships in canonical transformation.

Poincaré neither insisted that his original conditions of canonical transformation were given by 1-form nor proved them. However, historically, it is remarkable that he provided the origin of the modern condition. He gave proofs of relations (3.3-1) and (3.3-4), and he used them to develop his idea of canonical transformation in subsequent publications.

3.4 Poincaré's attitude in 1899

In volume 3 of *Les Méthodes Nouvelles* (1899), chapter 29 is devoted to a discussion of the principle of least action, which is the so-called Hamilton's principle of dynamics. Because Poincaré had written two papers that were related to the principle of least action in 1896–1897[11], he devoted one chapter to introducing related topics.

In his attention to the principle of least action, Poincaré provided proof of his characterization of the canonical of transformation by 1-form

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[11] "Sur les solutions périodiques et le principe de moindre action," *Comptes rendus*, CXXIII, 1896, pp.915-918, and "Les solutions périodiques et le principe de moindre action", *Comptes rendus*, CXXIV, 1897, pp.713-716.

(3.3.-4). His approach was different from the canonical transformation in volume 1 of *Les Méthodes Nouvelles.*

Poincaré introduced the so-called Hamilton's principle as follows

$$\delta J = \delta \int_{t_0}^{t_1} \left(-H + \sum p_i \frac{dq_i}{dt} \right) dt = 0. \quad (3.4\text{-}1)$$

He derived canonical equations of motion (1') from this principle in a way that differed from that shown in volume 1. In addition to the canonical equations, in this principle

$$\delta J = \left[\sum p_i \delta q_i \right]_{t=t_0}^{t=t_1} \quad (3.4\text{-}2)$$

should be 0 at the end points. We chose new variables $(Q_1, \cdots, Q_n, P_1, \cdots, P_n)$ such that

$$\sum P_i dQ_i - \sum p_i dq_i = dS \quad (3.4\text{-}3)$$

for some function S and set

$$J' = \int_{t_0}^{t_1} \left(-H + \sum P_i \frac{dQ_i}{dt} \right) dt \,.$$

Since

$$J' - J = \int \frac{dS}{dt} dt = S_{t_1} - S_{t_0}$$

then

$$\delta J' = \delta J + \left[\delta S \right]_{t=t_0}^{t=t_1} \quad \text{or} \quad \delta J' = \left[\sum P_i \delta Q_i \right]_{t=t_0}^{t=t_1} \qquad (3.4\text{-}4)$$

Principle (3.4-1) holds in dynamical systems that are written using the new variables. Poincaré then wrote that equation (3.4-4) is "equivalent" to equations (2') for the same reason that (3.4-2) is "equivalent" to (1') and equation (3.4-2) is "equivalent" to (3.4-4), and then equation (1') is "equivalent" to (2'); that is, the canonical form of equations is preserved if (3.4-3) holds.

In these descriptions, we find that Poincaré attained almost the same understanding that we have today. He gave the condition of the canonical transformation in terms of exact differential 1-form (3.4-3) and proved it based on Hamilton's principle. However, it is not Poincaré's final attitude toward canonical transformation. We continue to follow his arguments on canonical transformation.

In addition, in *Les Méthodes Nouvelles,* Poincaré described 1-form (3.3-1) in relation to invariant theory but not in relation to canonical transformation.

3.5 Poincaré's final comprehension of the canonical transformation

a) Poincaré's final characterization of canonical transformation

Different from his work in *Les Méthodes Nouvelles,* Poincaré discussed the practical usage of the theory of celestial mechanics in his three-volume *Leçons de Mécanique Céleste* (1905–1910). However, as in *Les Méthodes Nouvelles,* he discussed canonical transformation in the introductory part of *Leçons*. He wrote that the understanding of certain analytical transformations was indispensable in studying dynamics. Indeed, he devoted an entire chapter to explaining the theory of canonical transformation.

Poincaré first considered not equations of motion but any equation written in the canonical form:

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i},\quad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i},\quad H=H(x_1,\cdots,x_n,y_1,\cdots,y_n).\quad (3.5\text{-}1)$$

He set solutions to equations (3.5-1) as follows:

$$x_i=x_i(t,\alpha_1,\ldots,\alpha_n)\qquad y_i=y_i(t,\alpha_1,\ldots,\alpha_n),$$

where $\alpha_1,\ldots,\alpha_n$ are integral constants. By differentiating equations (3.5-1) Poincaré showed that they are equivalent to

$$\frac{d}{dt}\sum x_i\frac{\partial y}{\partial\alpha_k}-\frac{d}{d\alpha_k}\sum x_i\frac{\partial y_i}{\partial t}=\frac{\partial F}{\partial\alpha_k}.\quad (3.5\text{-}2)$$

He introduced new variables

$$X_i=X_i(x_1,\ldots,x_n,y_1\ldots,y_n),\ Y_i=Y_i(x_1,\ldots,x_n,y_1\ldots,y_n)\quad (i=1,\ldots,n)$$

and sought the transformation of variables $(x_1,\ldots,x_n,y_1\ldots,y_n)$ to new variables. Then he demonstrated that if the relationship

$$\sum X_i dY_i-\sum x_i dq_i=exact.,\quad (3\text{-}5.3)$$

holds, the new variables hold the relationship that corresponds to (3.5-2) by differentiating (3-5.3) with respect to $\alpha_k\ (k=1,\ldots,n)$ and t. Poincaré then concluded that canonical transformation occurs if the old and new variables are bounded by relationships (3-5.3); that is, the relationship he described in 1897 paper without demonstrating it.

Next, Poincaré introduced the canonical equations of motion as follows. Let $(X_1, X_2, X_3), \ldots, (X_{n-2}, X_{n-1}, X_n)$ be positions of $(n/3)$ mass points in orthogonal coordinates. Then kinematic energy is given as

$$T = \frac{1}{2}\sum m_i \left(\frac{dX_i}{dt}\right)^2,$$

and total energy is $H = T + U$, where U is potential energy, and equations of motions are given as

$$m_i \frac{dX_i^2}{dt^2} = -\frac{\partial U}{\partial X_i}.$$

Poincaré set $Y_i = m_i \frac{dX_i}{dt}$, noting that U depends only on X_i and T only on $\dot{X}_i$. He indicated that X_i, Y_i and H construct canonical equations of motion.

Poincaré defined curvilinear coordinates $q_1, \ldots, q_n$, which are generalized coordinates in modern terminology, as follows:

$$X_i = \psi_i(q_1, \ldots, q_n) \quad (3.5\text{-}4)$$

He differentiated (3.5-4), noting relationships among $X_i, \dot{X}_i$, $q_i, \dot{q}_i$ and setting

$$\frac{\partial T}{\partial \dot{q}_i} = p_i.$$

Since T is the second degree of the homogenous function of $\dot{X}_i$ or $\dot{p}_i$, Poincaré derived the following:

$$\sum p_i dq_i = \sum Y_i dX_i.$$

Therefore,

$$\sum q_i dp_i - \sum X_i dY_i = d\left[\sum p_i q_i - \sum X_i Y_i\right] \quad \text{or} \quad \sum q_i dp_i - \sum X_i dY_i = exact.$$

He then concluded that the transformation from (X_i, Y_i) to (q_i, p_i) was canonical.

b) Poincaré's version of Jacobi's method

Poincaré entitled section Jacobi's method. Let S be an unknown function and set $p_i = \frac{\partial S}{\partial q_i}$. Since total energy H was constant, he derived a non-linear first order differential equation, which is the so-called Hamilton–Jacobi equation:

$$H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}\right) = const. \quad (3.5\text{-}5)$$

He adopted a complete solution to equation (3.5-5) that involved arbitrary constants $\beta_1, \ldots, \beta_n$ as the "unknown function." The total energy H depends on these arbitrary constants, and (3.5-5) becomes

$$H\left(q_1, \cdots, q_n, \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_n}\right) = \varphi(\beta_1, \cdots, \beta_n).$$

In contrast, one obtains

$$dS = \sum \frac{\partial S}{\partial q_i} dq_i + \sum \frac{\partial S}{\partial \beta_i} d\beta_i$$

and

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial \beta_i} = \gamma_i$$

where $\gamma_1, \cdots, \gamma_n$ are new arbitrary constants. Poincaré noted that there are *2n* relationships among *4n* variables and decided that *2n* variables could be regarded as functions of another *2n* variable. For example, $(\beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n)$ could be regarded as functions of $(q_1, \cdots, q_n, p_1, \cdots, p_n)$. If one set the former as the new variable and the latter as the old variable, the following relationship holds:

$$\sum \gamma_i d\beta_i - \sum q_i dp_i = d(S - \sum q_i p_i)$$

Then the transformation is canonical. The canonical equations for q_i, p_i are transformed to

$$\frac{d\gamma_i}{dt} = \frac{\partial H}{\partial \beta_i} = \frac{\partial \varphi}{\partial \beta_i}, \quad \frac{d\beta_i}{dt} = -\frac{\partial H}{\partial \gamma_i} = 0. \quad (i = 1, \cdots, n) \quad (3.5\text{-}6)$$

Poincaré easily obtained the integral of equations (3.5-6) as follows:

$$\beta_i = const. \quad \gamma_i = \frac{\partial \varphi}{\partial \beta_i} t + \varpi_i$$

where $\varpi_1, \ldots, \varpi_n$ are integral constants. He noted that the solutions to the original equations written as (q_i, p_i) are obtained through (3.5-6). That is, he transformed the equations to facilitate their solutions. In other words, Poincaré solved canonical equations using the complete solution to the Hamilton–Jacobi equation.

Poincaré termed this procedure Jacobi's method. It is true that Jacobi had solved the canonical equations through a complete solution to the Hamilton–Jacobi equation. Consequently, mathematicians may find it difficult to call Poincaré' procedure Jacobi's method. However, Poincaré's procedure has a different aspect: one at first solves the Hamilton–Jacobi equation to transform the canonical equation to a simple one using a complete solution of the Hamilton–Jacobi equation. One then obtains the solution to original canonical equations through the solutions to simpler equations. Different from Jacobi's method, Poincaré's procedure involves the idea of canonical transformation.

It was Jacobi's idea to generate a function of the canonical transformation. However, he did not develop a complete solution to the generating function of the canonical transformation. However, Poincaré noted this relationship. Based on this understanding, Poincaré demonstrated his original method, calling it Jacobi's method.

Subsequently, Poincaré discussed the case where H explicitly depends on time, noting the 1-form condition and showing that function H is not retained in the transformed canonical equation.

In demonstrating Poincaré's version of Jacobi's method, Poincaré succeeded in demonstrating a complete solution, which became a generating function, which was the property he had suspected in 1892. However, based on our modern knowledge, Poincaré's demonstration was theoretically weak. He vaguely grasped that the generating function gives *2n* relations among *4n* old and new variables, and he did not examine it further. However, in 1907, Chariler distinguished four types of generating function:

$$S(\mathbf{q}, \mathbf{Q}), \quad S(\mathbf{q}, \mathbf{P}), \quad S(\mathbf{p}, \mathbf{Q}), \quad S(\mathbf{p}, \mathbf{P})$$

where $(\mathbf{q}, \mathbf{p})$ are old variables and $(\mathbf{Q}, \mathbf{P})$ are new variables. From Charlier's framework, Poincaré had chosen the case of $S(\mathbf{q}, \mathbf{P})$ and naturally had considered it the complete solution. Charlier's approach, although it was based on Jacobi's second theorem, was quite different from Poincaré's approach. Charlier's books contain ideas that were important in the development of the theory of canonical transformation. For example, he derived the origin of the action-angle variables, which Schwarzschild effectively used to explain the Stark effect in 1916. However, here we only indicate that his approach differed from Poincaré's approach, and we note his four types of the generating function.

4. Concluding Remarks: How Did Poincaré's Idea in 1899 Circulate Among Physicists?

Today we accept the results that Poincaré achieved in 1899 but not his final results for the following reason. Max Born's, influential textbook on mechanics, [12] at first adopted the same approach. Born derived the 1-form condition of (3.4-3) from Hamilton's principle. In fact, Born referred to Poincaré's *Les Méthodes Nouvelles* in his *Atommechanik.* Around 1916, physicists realized the Hamilton–Jacobi theory offered effective mathematical tools to the old quantum theory. Since the Hamilton–Jacobi theory had been developed in celestial mechanics, mainly German-speaking physicists had read and referred to Charlier's *Die Mechanik des Himmels* in studying this theory.

Compared with Charlier's books in German, Poincaré's books in the French language were not as popular. However, Born needed the mathematical method to analyze a degeneration system in discussing the

[12] *Vorlesungen über Atommechanik* , 1925, Julius Springer.

structure of the helium atom. In Göttingen, he read Poincaré's *Les Méthodes Nouvelles* with Heisenberg around 1922–1923.[13] However, Born's colleague David Hilbert, in his lectures entitled "Mathematical Foundation of Quantum Theory" (1922–1923), constructed a theory of geometrical optics and dynamics based on the variational principle.[14] In his lectures, Hilbert's mechanics is based on Hamilton's principle. Born then decided to construct his theory based on Hamilton's principle, and he chose Poincaré's 1899 approach. In due course, Born inserted Charlier's results, including the four types of generating functions. His description resembles that provided in modern textbooks.

Lothar Nordheim, Born's assistant, with Fues published an article[15] that introduced the basis of the Hamilton–Jacobi theory. Their introduction canonical transformation was almost the same as in Born's book.

The influential modern English textbook on mechanics, Herbert Goldstein's *Classical Mechanics*, refers to Nordheim and Fues's description of canonical transformations. Because of this process, Poincaré's 1899 approach should become the standard today.

[13] D. C. Cassidly, *Uncertainty: The life and Science of Werner Heisenberg*, 1992, W.H. Freewan & Company, Chapter 8.

[14] *D. Hilbert, Mathematische Grundlagen der Quantentheorie* (T. Sauer and U. Majer eds.) *David Hilbert's Lectures on the Foundations of Physics 1915-1927.* 2009, Springer, pp. 507-601.

[15] N. Nordheim and E.Fues, Die Hamilton-Jacobische Theorie der Dynamik" in *Handbuch der Physik* (1927), pp. 91-130.